

Recursive utility optimization with concave coefficients

Shaolin Ji *

Xiaomin Shi[†]

Abstract. This paper concerns the recursive utility maximization problem. We assume that the coefficients of the wealth equation and the recursive utility are concave. Then some interesting and important cases with nonlinear and nonsmooth coefficients satisfy our assumption. After given an equivalent backward formulation of our problem, we employ the Fenchel-Legendre transform and derive the corresponding variational formulation. By the convex duality method, the primal “sup-inf” problem is translated to a dual minimization problem and the saddle point of our problem is derived. Finally, we obtain the optimal terminal wealth. To illustrate our results, three cases for investors with ambiguity aversion are explicitly worked out under some special assumptions.

Key words. recursive utility optimization, convex duality, nonsmooth coefficient, nonlinear equation, saddle point

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1 Introduction

The problem of an agent who invests in a financial market so as to maximize the utility of his terminal wealth is a topic that is being widely studied. There have been many literatures on this issue with different viewpoints.

When the wealth equation is linear, the problem of maximizing the expected utility of terminal wealth is well understood in a complete or constrained financial market (refer to [4, 19, 21]). Since then many works are focused on this optimization problem with improved utility functions. For example, Bian [1] and Westray, Zheng [29], [30] studied this kind of problem with nonsmooth utility functions. In order to deal with model uncertainty, the so-called robust utility maximization problems are also investigated widely. Quenez [27] studied this problem in an incomplete multiple-priors model. Schied [28] explored the robust utility maximization problem in a complete market under the existence of a “least favorable measure”. Jin and Zhou [17] studied expected utility maximization problem as well as mean-variance problem when the appreciation rates are only known to be in a certain convex closed set. Other developments include the

*Institute for Financial Studies, Shandong University, Jinan 250100, China and Institute of Mathematics, Shandong University, Jinan 250100, China, Email: jsl@sdu.edu.cn. This work was supported by National Natural Science Foundation of China (No. 11571203); Supported by the Programme of Introducing Talents of Discipline to Universities of China (No.B12023).

[†]Corresponding author. Institute for Financial Studies, Shandong University, Jinan 250100, China. Email: shixm@mail.sdu.edu.cn, shixiaominhi@163.com. This work was supported by National Natural Science Foundation of China (No. 11401091).

recursive utility maximization problems which were investigated by Faïdi et al [8], Matoussi, Xing [23] and Epstein, Ji [11, 12].

But up to our knowledge, there are not many results related to this kind of optimization problem with nonlinear wealth equations. Cvitanic and Cuoco [3] explored the optimal consumption problem for a large investor whose portfolio strategies can affect the instantaneous expected returns of the assets. They show the existence of optimal policies by convex duality method developed in [4, 19]. Ji and Peng [14] studied the continuous time mean-variance problem with nonlinear wealth equation. El Karoui et al [9] obtained a dynamic maximum principle for the optimization of recursive utilities and characterized the optimal consumptions and portfolio strategies via a forward-backward SDE system. Their method depends heavily on the smoothness of the coefficients of the forward-backward SDE system.

As for the wealth equations, there are some interesting cases in which the coefficients are nonlinear and nonsmooth. As shown in [3], the wealth equation of a large investor may be nonsmooth. The other well-known case is that an investor is allowed to borrow money with a higher interest rate. As for the recursive utilities, some important generators are also nonsmooth, for instance, the K-ignorance case which was proposed by Chen and Epstein [2]. The coefficients of the wealth equations and the recursive utilities of the above cases are all concave. This motivate us to study the recursive utility maximization problem with concave coefficients of both the wealth equations and the recursive utilities in this paper.

We first give an equivalent backward formulation of our problem. This “backward formulation” was introduced by El Karoui, Peng and Quenez [9] in order to solve a recursive utility optimization problem, and employed by Ji and Peng [14] to obtain a necessary condition for a mean-variance portfolio selection problem with non-convex wealth equations. For its application in stochastic control with state constraints, we refer the reader to Ji, Zhou [15, 16]. El Karoui, Peng and Quenez [9] took the terminal wealth as the control variable, and then used a variational technique to obtain a stochastic maximum principle, i.e., a first-order necessary condition that characterizes the terminal wealth. In our context, we still take the terminal wealth as the control variable. But the stochastic maximum principle approach does not work due to the nonsmoothness of the coefficients. In order to overcome this difficulty, we assume that the coefficients are concave and derive a variational formulation by the Fenchel-Legendre transform of the coefficients which leads to a stochastic game problem. Inspired by the convex duality method developed in Cvitanic and Karatzas [5], we turn the primal “sup-inf” problem to a dual minimization problem and the saddle point of this game is derived. Then we obtain the optimal terminal wealth and the optimal portfolio process can be derived by the martingale representation theorem.

Three cases for investors with ambiguity aversion are provided to show the applications of our results. In these cases, we specialize the generator of the recursive utility as the K-ignorance case and the utility function of the terminal wealth as $u(x) = \frac{1}{\alpha}x^\alpha$, $0 < \alpha < 1$. By the main results in section 4, we characterize the saddle point via a quadratic BSDE and obtain the optimal terminal wealth explicitly. Especially, for the large investor case, we work out the explicit saddle point, the optimal wealth process, the optimal portfolio strategies as well as the utility intensity process.

This paper is organized as follows. In section 2, we give the classical, backward and variational formulation of the recursive utility maximization problem. Our main results are obtained in section 3. In section 4, we study three cases in which the investors are assumed to be ambiguity aversion (K-ignorance). The saddle

point and the optimal terminal wealth for each case are derived explicitly.

2 Formulation of the problem

In this paper, we study the recursive utility maximization problem with bankruptcy prohibition.

2.1 The wealth process

Let $W = (W^1, \dots, W^d)'$ be a standard d -dimensional Brownian motion defined on a filtered complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$, where $\{\mathcal{F}_t\}_{t \geq 0}$ denotes the natural filtration associated with the d -dimensional Brownian motion W and augmented.

Consider a financial market consisting of a riskless asset (the money market instrument or bond) whose price is S^0 and d risky securities (the stocks) whose prices are S^1, \dots, S^d . An investor decides at time $t \in [0, T]$ what amount π_t^i of his wealth X_t to invest in the i th stock, $i = 1, \dots, d$. The portfolio $\pi_t = (\pi_t^1, \dots, \pi_t^d)'$ and $\pi_t^0 := X_t - \sum_{i=1}^d \pi_t^i$ are \mathcal{F}_t -adapted. We suppose that the wealth process $X_t \equiv X_t^{x, \pi}$ of the investor who is endowed with initial wealth $x > 0$ is governed by the following stochastic differential equation,

$$\begin{cases} dX_t = b(t, X_t, \sigma_t' \pi_t) dt + \pi_t' \sigma_t dW_t; \\ X_0 = x, \end{cases} \quad (2.1)$$

where b is a given function and the predictable and invertible process $\sigma_t = \{\sigma_t^{ij}\}_{1 \leq i, j \leq d}$ is the stock volatility. σ_t is assumed to be bounded, uniformly in $(t, \omega) \in [0, T] \times \Omega$, and $\exists \varepsilon > 0$, $\rho' \sigma_t \sigma_t' \rho \geq \varepsilon \|\rho\|^2$, $\forall \rho \in \mathbb{R}^d$, $t \in [0, T]$, *a.s.*

Example 2.1 *The standard linear case.*

The price processes S_t^0 and S_t^1, \dots, S_t^d are governed by

$$\begin{aligned} dS_t^0 &= S_t^0 r_t dt, \quad S_0^0 = s_0; \\ dS_t^i &= S_t^i [b_t^i dt + \sum_{j=1}^d \sigma_t^{ij} dW_t^j], \quad S_0^i = s_i > 0; \quad i = 1, \dots, d. \end{aligned} \quad (2.2)$$

All processes r_t , $b_t = (b_t^1, \dots, b_t^d)'$, $\sigma_t = \{\sigma_t^{ij}\}_{1 \leq i, j \leq d}$, σ_t^{-1} are assumed to be predictable and bounded, uniformly in $(t, \omega) \in [0, T] \times \Omega$. Then the wealth process X_t satisfies the following linear stochastic differential equation,

$$\begin{cases} dX_t = (r_t X_t + \pi_t' (b_t - r_t \mathbf{1})) dt + \pi_t' \sigma_t dW_t, \\ X_0 = x, \end{cases}$$

where $\mathbf{1}$ is the d -dimensional vector whose every component is 1. In this example,

$$b(t, X_t, \sigma_t' \pi_t) = r_t X_t + \pi_t' (b_t - r_t \mathbf{1}). \quad (2.3)$$

Example 2.2 *The borrowing rate R_t is higher than the risk-free rate r_t .*

The stock prices are (2.2). Now the borrowing rate R_t is higher than the risk-free rate r_t , i.e., $R_t \geq r_t$, $t \in [0, T]$, a.s. In this case, the wealth process becomes

$$\begin{cases} dX_t = (r_t X_t + \pi'_t(b_t - r_t \mathbf{1}) - (R_t - r_t)(X_t - \pi'_t \mathbf{1})^-)dt + \pi'_t \sigma_t dW_t, \\ X_0 = x. \end{cases}$$

In this example,

$$b(t, X_t, \sigma'_t \pi_t) = r_t X_t + \pi'_t(b_t - r_t \mathbf{1}) - (R_t - r_t)(X_t - \pi'_t \mathbf{1})^-. \quad (2.4)$$

Example 2.3 A large investor case.

Cuoco and Cvitanic [3] considered the optimal portfolio choice problem for a large investor whose portfolio strategies can affect the price processes of the securities. In [3], the price processes are given by

$$\begin{aligned} dS_t^0 &= S_t^0 [r_t + l_0(X_t, \pi_t)]dt, \quad S_0^0 = s_0; \\ dS_t^i &= S_t^i \left[(b_t^i + l_i(X_t, \pi_t))dt + \sum_{j=1}^d \sigma_t^{ij} dW_t^j \right], \quad S_0^i = s_i > 0, \quad i = 1, \dots, d, \end{aligned}$$

where $l_i : \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}$, $i = 0, 1, \dots, d$ are some given functions which describe the effect of the wealth and the strategies possessed by the large investor. The wealth process is governed by

$$\begin{cases} dX_t = (r_t X_t + (X_t - \pi'_t \mathbf{1})l_0(X_t, \pi_t) + \pi'_t[b_t - r_t \mathbf{1} + l(X_t, \pi_t)])dt + \pi'_t \sigma_t dW_t, \\ X_0 = x. \end{cases}$$

In this example,

$$b(t, X_t, \sigma'_t \pi_t) = r_t X_t + (X_t - \pi'_t \mathbf{1})l_0(X_t, \pi_t) + \pi'_t[b_t - r_t \mathbf{1} + l(X_t, \pi_t)].$$

Cuoco and Cvitanic [3] also gave the following more specific example. For $x \in \mathbb{R}$,

$$\text{sgn}(x) := \begin{cases} \frac{|x|}{x}, & \text{if } x \neq 0; \\ 0, & \text{otherwise.} \end{cases} \quad (2.5)$$

Set $\varepsilon = (\varepsilon_1, \dots, \varepsilon_d)'$, $\text{sgn}(\pi_t) = (\text{sgn}(\pi_t^1), \dots, \text{sgn}(\pi_t^d))'$ and $\varepsilon \otimes \text{sgn}(\pi_t) = (\varepsilon_1 \text{sgn}(\pi_t^1), \dots, \varepsilon_d \text{sgn}(\pi_t^d))'$ where ε_i are given small positive numbers.

Consider $l_0(X_t, \pi_t) = 0$ and $l_i(X_t, \pi_t) = -\varepsilon_i \text{sgn}(\pi_t^i)$, $i = 1, \dots, d$. Then we have

$$b(t, X_t, \sigma'_t \pi_t) = r_t X_t + \pi'_t(b_t - r_t \mathbf{1} - \varepsilon \otimes \text{sgn}(\pi_t)). \quad (2.6)$$

For this specific large investor model, longing the i th risky security depresses its expected return while shorting it increases its expected return as explained in Cuoco and Cvitanic [3].

We introduce the following spaces:

$$\begin{aligned}
L^2(\Omega, \mathcal{F}_T, P; R) &= \left\{ \xi : \Omega \rightarrow R \mid \xi \text{ is } \mathcal{F}_T\text{-measurable, and } E|\xi|^2 < \infty \right\}, \\
M^2(0, T; R^d) &= \left\{ \phi : [0, T] \times \Omega \rightarrow R^d \mid (\phi_t)_{0 \leq t \leq T} \text{ is } \{\mathcal{F}_t\}_{t \geq 0}\text{-progressively measurable process,} \right. \\
&\quad \left. \text{and } \|\phi\|^2 = E \int_0^T |\phi_t|^2 dt < \infty \right\}, \\
S^2(0, T; R^d) &= \left\{ \phi : [0, T] \times \Omega \rightarrow R \mid (\phi_t)_{0 \leq t \leq T} \text{ is } \{\mathcal{F}_t\}_{t \geq 0}\text{-progressively measurable process,} \right. \\
&\quad \left. \text{and } \|\phi\|_S^2 = E \left[\sup_{0 \leq t \leq T} |\phi_t|^2 \right] < \infty \right\}.
\end{aligned}$$

For notational simplicity, we will often write L^2 , M^2 and S^2 instead of $L^2(\Omega, \mathcal{F}_T, P; R)$, $M^2(0, T; R^d)$ and $S^2(0, T; R)$ respectively.

Let $b(\omega, t, X, \pi) : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$.

Assumption 2.4 $b(\omega, t, X, \pi) : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ is $\{\mathcal{F}_t\}_{t \geq 0}$ -progressively measurable for any $(X, \pi) \in \mathbb{R} \times \mathbb{R}^d$ and

(i) There exists a constant $C_1 \geq 0$ such that

$$|b(\omega, t, X^1, \pi^1) - b(\omega, t, X^2, \pi^2)| \leq C_1(|X^1 - X^2| + |\pi^1 - \pi^2|), \quad \forall (t, \omega, X^1, X^2, \pi^1, \pi^2) \in \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d.$$

(ii) $b(t, 0, 0) \geq 0$, $t \in [0, T]$, a.s. and $E \int_0^T b^2(t, 0, 0) dt < +\infty$.

(iii) The function $(X, \pi) \mapsto b(\omega, t, X, \pi)$ is concave for all $(\omega, t) \in \Omega \times [0, T]$.

2.2 The recursive utility

In the time-additive expected utility maximization models, one can not separate the risk aversion and intertemporal substitution. To overcome this intertwine, Duffie and Epstein [6] introduced the stochastic recursive utility in the continuous time. In El Karoui, Peng and Queuenz [10], the stochastic recursive utility can be formulated in a more general form by backward stochastic differential equation (BSDE for short):

$$Y_t = u(X_T) + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z'_s dW_s. \quad (2.7)$$

We need the following assumptions.

Assumption 2.5 $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ is $\{\mathcal{F}_t\}_{t \geq 0}$ -progressively measurable for any $(y, z) \in \mathbb{R} \times \mathbb{R}^d$, and satisfies

(i) There exists a constant $C \geq 0$ such that

$$|f(\omega, t, Y_1, Z_1) - f(\omega, t, Y_2, Z_2)| \leq C(|Y_1 - Y_2| + |Z_1 - Z_2|), \quad \forall (\omega, t, Y_1, Y_2, Z_1, Z_2) \in \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d;$$

(ii) f is continuous in t and $E \int_0^T f^2(t, 0, 0) dt < +\infty$;

(iii) The function $(Y, Z) \mapsto f(\omega, t, Y, Z)$ is concave for all $(\omega, t) \in \Omega \times [0, T]$.

Assumption 2.6 $u : (0, \infty) \rightarrow \mathbb{R}$ is strictly increasing, strictly concave and of class C^2 , and satisfies
(i) Inada condition:

$$u'(0+) := \lim_{x \rightarrow 0+} u'(x) = \infty, u'(\infty) := \lim_{x \rightarrow \infty} u'(x) = 0;$$

(ii) $\exists k_1, k_2 \geq 0, p \in (0, 1)$ such that $|u(x)| \leq k_1 + k_2 \frac{x^p}{p}$;

(iii) $u(0) := \lim_{x \rightarrow 0+} u(x) > -\infty; u(\infty) := \lim_{x \rightarrow \infty} u(x) = \infty$.

2.3 Classical formulation

We consider that an investor chooses a portfolio strategy so as to

$$\begin{aligned} & \text{Maximize } Y_0^{x, \pi}, \\ & s.t. \begin{cases} X_t \geq 0, \\ \pi \in M^2, \\ (X, \pi) \text{ satisfies (2.1),} \\ (Y, Z) \text{ satisfies (2.7),} \end{cases} \end{aligned} \tag{2.8}$$

where $X_t \geq 0$ describes that no-bankruptcy is required.

Definition 2.7 A portfolio π is said to be admissible if $\pi \in M^2$ and the corresponding wealth process $X_t \geq 0, t \in [0, T], a.s.$

Given the initial wealth $x > 0$, denote by $\bar{\mathcal{A}}(x)$ the set of an investor's admissible portfolio strategies, that is

$$\bar{\mathcal{A}}(x) \equiv \bar{\mathcal{A}}(x; 0, T) = \left\{ \pi \mid \pi \in M^2, X_t^{x, \pi} \geq 0, t \in [0, T], a.s. \right\}. \tag{2.9}$$

Thus, (2.8) can be written as:

$$\begin{aligned} & \text{Maximize } Y_0^{x, \pi}, \\ & s.t. \begin{cases} \pi \in \bar{\mathcal{A}}(x), \\ (X, \pi) \text{ satisfies (2.1),} \\ (Y, Z) \text{ satisfies (2.7),} \end{cases} \end{aligned}$$

2.4 Backward formulation

In this subsection, we give an equivalent backward formulation of the above optimization problem (2.8). This backward formulation is founded in [9, 14–16].

Set

$$\begin{aligned} q_t &= \sigma_t' \pi_t, \text{ a.s., } \forall t \in [0, T], \\ U &= \{\xi \mid \xi \in L^2 \text{ and } \xi \geq 0\}. \end{aligned}$$

Since σ_t is invertible, q_t can be regarded as the control variable instead of π_t . Notice that selecting q is equivalent to selecting the terminal wealth X_T by the existence and uniqueness result of BSDEs (refer to Theorem 2.1 in [10]). Hence the wealth equation (2.1) can be rewritten as

$$\begin{cases} -dX_t = -b(t, X_t, q_t)dt - q'_t dW_t, \\ X_T = \xi, \end{cases} \quad (2.10)$$

where the terminal wealth ξ is the “control” to be chosen from U . Note that we will require that the solution X of (2.10) at time 0 equals the initial wealth x .

If we take the terminal wealth as control variable, the recursive utility process can be written as:

$$\begin{cases} -dY_t = f(t, Y_t, Z_t)dt - Z'_t dW_t, \\ Y_T = u(\xi). \end{cases} \quad (2.11)$$

Assumption 2.6 guarantees that $u(\xi) \in L^2$ for any $\xi \in U$. By the existence and uniqueness result of BSDEs, we know that for any $\xi \in U$, there exists a unique solution (X_t, q_t) (resp. (Y_t, Z_t)) of (2.10) (resp. (2.11)). By the comparison theorem of BSDEs, Assumption 2.4 and the nonnegative terminal wealth keeps the wealth process be nonnegative all the time. Usually, we denote the solution Y of (2.11) at time 0 by Y_0^ξ .

This gives rise to the following optimization problem:

$$\begin{aligned} & \text{Maximize } J(\xi) := Y_0^\xi, \\ & s.t. \begin{cases} \xi \in U, \\ X_0 = x, \\ (X, q) \text{ and } (Y, Z) \text{ satisfy (2.10) and (2.11) respectively.} \end{cases} \end{aligned} \quad (2.12)$$

Definition 2.8 *A random variable $\xi \in U$ is called feasible for the initial wealth x if and only if $X_0 = x$. We will denote by $\mathcal{A}(x)$ the set of all feasible ξ for the initial wealth x .*

It is clear that the original problem (2.8) is equivalent to the auxiliary one (2.12). Hence, hereafter we focus ourselves on solving problem (2.12). The advantage of doing this is that, since ξ is the control variable, the state constraint in (2.8) becomes a control constraint in (2.12), whereas it is well known in control theory that a control constraint is much easier to deal with than a state constraint. The cost of this approach is that the original initial condition $X_0 = x$ becomes a constraint.

2.5 Variational formulation

Let $\tilde{b}(\omega, t, \mu, \nu)$ be the Fenchel-Legendre transform of b :

$$\tilde{b}(\omega, t, \mu, \nu) = \sup_{(x, q) \in \mathbb{R} \times \mathbb{R}^d} [b(\omega, t, x, q) - x\mu - q'\nu], \quad (\mu, \nu) \in \mathbb{R} \times \mathbb{R}^d. \quad (2.13)$$

The effective domain of \tilde{b} is

$$\mathcal{D}_{\tilde{b}} := \{(\omega, t, \mu, \nu) \in \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \mid \tilde{b}(\omega, t, \mu, \nu) < +\infty\}.$$

As was shown in [10], the (ω, t) -section of $\mathcal{D}_{\tilde{b}}$, denoted by $\mathcal{D}_{\tilde{b}}^{(\omega, t)}$ is included in the bounded domain

$$B' := [-C_1, C_1]^{d+1} \subset \mathbb{R} \times \mathbb{R}^d,$$

where C_1 is the Lipschitz constant of b . The following duality relation is due to the concavity of b ,

$$b(\omega, t, x, q) = \inf_{(\mu, \nu) \in \mathcal{D}'_{\tilde{b}}^{(\omega, t)}} [\tilde{b}(\omega, t, \mu, \nu) + x\mu + q'\nu]. \quad (2.14)$$

Set

$$\mathcal{B}' = \{(\mu, \nu) \mid (\mu, \nu) \text{ is } \{\mathcal{F}_t\}_{t \geq 0}\text{-progressively measurable and } B'\text{-valued and } E \int_0^T \tilde{b}(\omega, t, \mu_t, \nu_t)^2 dt < +\infty\}.$$

Let $F(\omega, t, \beta, \gamma)$ be the Fenchel-Legendre transform of f :

$$F(\omega, t, \beta, \gamma) = \sup_{(y, z) \in \mathbb{R} \times \mathbb{R}^d} [f(\omega, t, y, z) - y\beta - z'\gamma], \quad (\beta, \gamma) \in \mathbb{R} \times \mathbb{R}^d. \quad (2.15)$$

The effective domain of F is

$$\mathcal{D}_F := \{(\omega, t, \beta, \gamma) \in \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \mid F(\omega, t, \beta, \gamma) < +\infty\}.$$

Similarly, the (ω, t) -section of \mathcal{D}_F , denoted by $\mathcal{D}_F^{(\omega, t)}$ is included in the bounded domain

$$B := [-C, C]^{d+1} \subset \mathbb{R} \times \mathbb{R}^d,$$

where C is the Lipschitz constant of f . We have the duality relation by the concavity of f ,

$$f(\omega, t, y, z) = \inf_{(\beta, \gamma) \in \mathcal{D}_F^{(\omega, t)}} [F(\omega, t, \beta, \gamma) + y\beta + z'\gamma]. \quad (2.16)$$

Set

$$\mathcal{B} = \{(\beta, \gamma) \mid (\beta, \gamma) \text{ is } \{\mathcal{F}_t\}_{t \geq 0}\text{-progressively measurable and } B\text{-valued and } E \int_0^T F(t, \beta_t, \gamma_t)^2 dt < +\infty\}.$$

Assumption 2.9 *The functions \tilde{b} and F are bounded on their effective domains.*

Lemma 2.10 *Under Assumption 2.4, 2.5 and 2.9, \mathcal{B}' and \mathcal{B} are convex sets and are closed under almost sure convergence.*

Proof: The convexity of \mathcal{B}' and \mathcal{B} comes from the convexity of \tilde{b} and F . Assumption 2.9 with Fatou's Lemma guarantees the closeness. \square

For any $(\mu, \nu) \in \mathcal{B}'$, define

$$b^{\mu, \nu}(t, y, z) = \tilde{b}(t, \mu_t, \nu_t) + y\mu_t + \pi'\nu_t,$$

and denote by $(X^{\mu, \nu}, q^{\mu, \nu})$ the unique solution to the linear BSDE (2.10) associated to $b^{\mu, \nu}$. For any $(\beta, \gamma) \in \mathcal{B}$, define

$$f^{\beta, \gamma}(t, y, z) = F(t, \beta_t, \gamma_t) + y\beta_t + z'\gamma_t,$$

and denote by $(Y^{\beta, \gamma}, Z^{\beta, \gamma})$ the unique solution to the linear BSDE (2.11) associated to $f^{\beta, \gamma}$.

For $0 \leq t \leq s \leq T$, set

$$\begin{aligned} N_{t,s}^{\mu,\nu} &= e^{-\int_t^s (\mu_r + \frac{1}{2}\|\nu_r\|^2)dr - \int_t^s \nu_r' dW_r}, \\ \Gamma_{t,s}^{\beta,\gamma} &= e^{\int_t^s (\beta_r - \frac{1}{2}\|\gamma_r\|^2)dr + \int_t^s \gamma_r' dW_r}. \end{aligned}$$

By the method similar to Proposition 3.4 in [10], we have the following variational formulation of X_t and Y_t .

Under Assumption 2.4 and 2.5, the solutions (X_t, q_t) and (Y_t, Z_t) of (2.10) and (2.11) can be represented as

$$\begin{aligned} X_t &= \operatorname{ess\,sup}_{(\mu,\nu) \in \mathcal{B}'} X_t^{\mu,\nu}, \quad a.s., \\ Y_t &= \operatorname{ess\,inf}_{(\beta,\gamma) \in \mathcal{B}} Y_t^{\beta,\gamma}, \quad a.s., \end{aligned}$$

where

$$\begin{aligned} X_t^{\mu,\nu} &= E\left[-\int_t^T N_{t,s}^{\mu,\nu} \tilde{b}(s, \mu, \nu) ds + N_{t,T}^{\mu,\nu} \xi | \mathcal{F}_t\right], \\ Y_t^{\beta,\gamma} &= E\left[\int_t^T \Gamma_{t,s}^{\beta,\gamma} F(s, \beta_s, \gamma_s) ds + \Gamma_{t,T}^{\beta,\gamma} u(\xi) | \mathcal{F}_t\right], \end{aligned}$$

Especially, we have

$$\begin{aligned} X_0 &= \sup_{(\mu,\nu) \in \mathcal{B}'} E\left[-\int_0^T N_{0,s}^{\mu,\nu} \tilde{b}(s, \mu_s, \nu_s) ds + N_{0,T}^{\mu,\nu} \xi\right], \\ Y_0 &= \inf_{(\beta,\gamma) \in \mathcal{B}} E\left[\int_0^T \Gamma_{0,s}^{\beta,\gamma} F(s, \beta_s, \gamma_s) ds + \Gamma_{0,T}^{\beta,\gamma} u(\xi)\right]. \end{aligned}$$

By the above analysis,

$$\mathcal{A}(x) = \{\xi \in U \mid \sup_{(\mu,\nu) \in \mathcal{B}'} E\left[-\int_0^T N_{0,s}^{\mu,\nu} \tilde{b}(s, \mu_s, \nu_s) ds + N_{0,T}^{\mu,\nu} \xi\right] = x\}. \quad (2.17)$$

Now our problem (2.12) is equivalent to the following problem:

$$\begin{aligned} \text{Maximize } J(\xi) &= Y_0^\xi = \inf_{(\beta,\gamma) \in \mathcal{B}} E\left[\int_0^T \Gamma_{0,s}^{\beta,\gamma} F(s, \beta_s, \gamma_s) ds + \Gamma_{0,T}^{\beta,\gamma} u(\xi)\right], \\ \text{s.t. } \xi &\in \mathcal{A}(x). \end{aligned} \quad (2.18)$$

It is essentially a robust optimization problem. Define the “max-min” quantity

$$\underline{V}(x) = \sup_{\xi \in \mathcal{A}(x)} \inf_{(\beta,\gamma) \in \mathcal{B}} E\left[\int_0^T \Gamma_{0,s}^{\beta,\gamma} F(s, \beta_s, \gamma_s) ds + \Gamma_{0,T}^{\beta,\gamma} u(\xi)\right]. \quad (2.19)$$

It is dominated by its “min-max” counterpart

$$\bar{V}(x) = \inf_{(\beta,\gamma) \in \mathcal{B}} \sup_{\xi \in \mathcal{A}(x)} E\left[\int_0^T \Gamma_{0,s}^{\beta,\gamma} F(s, \beta_s, \gamma_s) ds + \Gamma_{0,T}^{\beta,\gamma} u(\xi)\right].$$

If we can find $(\hat{\beta}, \hat{\gamma}, \hat{\xi}) \in \mathcal{B} \times \mathcal{A}(x)$ such that

$$\underline{V}(x) = E\left[\int_0^T \Gamma_{0,s}^{\hat{\beta}, \hat{\gamma}} F(s, \hat{\beta}_s, \hat{\gamma}_s) ds + \Gamma_{0,T}^{\hat{\beta}, \hat{\gamma}} u(\hat{\xi})\right] = \bar{V}(x), \quad (2.20)$$

then the optimal solution of problem (2.18) is $\hat{\xi}$.

3 Main results

Denote the inverse of the marginal utility function $u'(\cdot)$ by $I(\cdot)$. The convex dual

$$\tilde{u}(\zeta) := \max_{x>0} [u(x) - \zeta x] = u(I(\zeta)) - \zeta I(\zeta), \quad \zeta > 0. \quad (3.1)$$

For $0 < \zeta < \infty$, introduce the value functions

$$\tilde{V}(\zeta) = \inf_{\substack{(\beta, \gamma) \in \mathcal{B} \\ (\mu, \nu) \in \mathcal{B}'}} E \left[\int_0^T (\Gamma_{0,s}^{\beta, \gamma} F(s, \beta_s, \gamma_s) + \zeta N_{0,s}^{\mu, \nu} \tilde{b}(s, \mu_s, \nu_s)) ds + \Gamma_{0,T}^{\beta, \gamma} \tilde{u}(\zeta \frac{N_{0,T}^{\mu, \nu}}{\Gamma_{0,T}^{\beta, \gamma}}) \right] \quad (3.2)$$

and

$$\begin{aligned} V_*(x) &= \inf_{\substack{(\beta, \gamma) \in \mathcal{B} \\ (\mu, \nu) \in \mathcal{B}' \\ \zeta > 0}} E \left[\int_0^T (\Gamma_{0,s}^{\beta, \gamma} F(s, \beta_s, \gamma_s) + \zeta N_{0,s}^{\mu, \nu} \tilde{b}(s, \mu_s, \nu_s)) ds + \Gamma_{0,T}^{\beta, \gamma} \tilde{u}(\zeta \frac{N_{0,T}^{\mu, \nu}}{\Gamma_{0,T}^{\beta, \gamma}}) + \zeta x \right] \\ &= \inf_{\zeta > 0} [\tilde{V}(\zeta) + \zeta x]. \end{aligned} \quad (3.3)$$

Lemma 3.1 *Under Assumption 2.5, 2.6 and 2.9, for any given $\zeta > 0$, there exist pairs $(\hat{\beta}, \hat{\gamma}) = (\hat{\beta}_\zeta, \hat{\gamma}_\zeta) \in \mathcal{B}$ and $(\hat{\mu}, \hat{\nu}) = (\hat{\mu}_\zeta, \hat{\nu}_\zeta) \in \mathcal{B}'$ which attain the infimum in (3.2).*

Proof: By the boundedness of \mathcal{B} and \mathcal{B}' , the sets $\tilde{\mathcal{B}}' = \{N_{0,T}^{\mu, \nu} : (\mu, \nu) \in \mathcal{B}'\}$ and $\tilde{\mathcal{B}} = \{\Gamma_{0,T}^{\beta, \gamma} : (\beta, \gamma) \in \mathcal{B}\}$ are bounded in L^p for any $p \geq 1$. So $(\mu, \nu) \in \mathcal{B}'$ (resp. $(\beta, \gamma) \in \mathcal{B}$) is uniquely determined by $N_{0,T}^{\mu, \nu} \in \tilde{\mathcal{B}}'$ (resp. $\Gamma_{0,T}^{\beta, \gamma} \in \tilde{\mathcal{B}}$) (up to a.e. a.s. equivalence). Then,

$$\tilde{V}(\zeta) = \inf_{\substack{\Gamma_{0,T}^{\beta, \gamma} \in \tilde{\mathcal{B}} \\ N_{0,T}^{\mu, \nu} \in \tilde{\mathcal{B}}'}} E \left[\int_0^T (\Gamma_{0,s}^{\beta, \gamma} F(s, \beta_s, \gamma_s) + \zeta N_{0,s}^{\mu, \nu} \tilde{b}(s, \mu_s, \nu_s)) ds + \Gamma_{0,T}^{\beta, \gamma} \tilde{u}(\zeta \frac{N_{0,T}^{\mu, \nu}}{\Gamma_{0,T}^{\beta, \gamma}}) \right], \quad 0 < \zeta < \infty.$$

Note that the function

$$(x, y) \mapsto x \tilde{u}(\zeta \frac{y}{x}), \quad \zeta > 0$$

is convex (not strictly). Then, by the method similar to that in [3] (Theorem 3), we obtain the existence of $(\hat{\beta}, \hat{\gamma})$ and $(\hat{\mu}, \hat{\nu})$. \square

Lemma 3.2 *Under Assumption 2.4, 2.5 and 2.6, for any given $x > 0$, there exists a number $\hat{\zeta}_x \in (0, \infty)$ which attains the infimum of $V_*(x) = \inf_{\zeta > 0} [\tilde{V}(\zeta) + \zeta x]$.*

Proof: By Assumption 2.4, we know that for any $s \in [0, T]$ and $(\mu, \nu) \in \mathcal{B}'$, $\tilde{b}(s, \mu_s, \nu_s) \geq 0$, a.s. Then, $\forall (\beta, \gamma) \in \mathcal{B}$ and $(\mu, \nu) \in \mathcal{B}'$,

$$\begin{aligned} &E \left[\int_0^T (\Gamma_{0,s}^{\beta, \gamma} F(s, \beta_s, \gamma_s) + \zeta N_{0,s}^{\mu, \nu} \tilde{b}(s, \mu_s, \nu_s)) ds + \Gamma_{0,T}^{\beta, \gamma} \tilde{u}(\zeta \frac{N_{0,T}^{\mu, \nu}}{\Gamma_{0,T}^{\beta, \gamma}}) \right] \\ &\geq E \left[\int_0^T \Gamma_{0,s}^{\beta, \gamma} F(s, \beta_s, \gamma_s) ds + \Gamma_{0,T}^{\beta, \gamma} \tilde{u}(\zeta \frac{N_{0,T}^{\mu, \nu}}{\Gamma_{0,T}^{\beta, \gamma}}) \right]. \end{aligned}$$

By Assumption 2.9, there exists a constant M_1 such that $\forall(\beta, \gamma) \in \mathcal{B}$,

$$E \int_0^T \Gamma_{0,s}^{\beta,\gamma} F(s, \beta_s, \gamma_s) ds \geq M_1.$$

Due to the monotonicity of \tilde{u} and the convexity of $(x, y) \mapsto x\tilde{u}(\zeta \frac{y}{x})$, we deduce the following inequality by Jensen's inequality

$$E[\Gamma_{0,T}^{\beta,\gamma} \tilde{u}(\zeta \frac{N_{0,T}^{\mu,\nu}}{\Gamma_{0,T}^{\beta,\gamma}})] \geq M_2 \tilde{u}(M_3 \zeta), \quad \forall(\beta, \gamma) \in \mathcal{B} \text{ and } (\mu, \nu) \in \mathcal{B}',$$

where the constants $M_2 > 0$, $M_3 > 0$ depend on the bound of F and the Lipschitz constants of b, f .

Hence $\tilde{V}(\zeta) \geq M_1 + M_2 \tilde{u}(M_3 \zeta)$, $\forall \zeta > 0$. By Assumption 2.6,

$$\tilde{V}(0) := \lim_{\zeta \rightarrow 0^+} \tilde{V}(\zeta) \geq M_1 + M_2 \lim_{\zeta \rightarrow 0^+} \tilde{u}(M_3 \zeta) = M_1 + M_2 u(\infty) = \infty. \quad (3.4)$$

From Lemma 4.2 in [20] and Assumption 2.6, we know

$$\tilde{u}(\infty) := \lim_{\zeta \rightarrow \infty} \tilde{u}(\zeta) = u(0) > -\infty.$$

Then we have

$$\lim_{\zeta \rightarrow \infty} [\tilde{V}(\zeta) + \zeta x] \geq \lim_{\zeta \rightarrow \infty} [M_1 + M_2 \tilde{u}(M_3 \zeta) + \zeta x] = \infty, \quad \forall x > 0. \quad (3.5)$$

So there exists a number $\hat{\zeta}_x \in (0, \infty)$ which attains the infimum of $V_*(x)$. \square

Lemma 3.3 *Under Assumption 2.4, 2.5, 2.6 and 2.9,*

$$V_*(x) = E\left[\int_0^T (\Gamma_{0,s}^{\hat{\beta}, \hat{\gamma}} F(s, \hat{\beta}_s, \hat{\gamma}_s) + \hat{\zeta} N_{0,s}^{\hat{\mu}, \hat{\nu}} \tilde{b}(s, \hat{\mu}_s, \hat{\nu}_s)) ds + \Gamma_{0,T}^{\hat{\beta}, \hat{\gamma}} \tilde{u}\left(\hat{\zeta} \frac{N_{0,T}^{\hat{\mu}, \hat{\nu}}}{\Gamma_{0,T}^{\hat{\beta}, \hat{\gamma}}}\right) + \hat{\zeta} x\right]$$

with $\hat{\zeta} = \hat{\zeta}_x$ as in lemma 3.2 and $(\hat{\beta}, \hat{\gamma}) = (\hat{\beta}_{\hat{\zeta}}, \hat{\gamma}_{\hat{\zeta}}) \in \mathcal{B}$, $(\hat{\mu}, \hat{\nu}) = (\hat{\mu}_{\hat{\zeta}}, \hat{\nu}_{\hat{\zeta}}) \in \mathcal{B}'$ as in lemma 3.1.

Proof: We have $\forall(\beta, \gamma) \in \mathcal{B}$, $\forall(\mu, \nu) \in \mathcal{B}'$, $\forall \zeta \in (0, \infty)$, $\forall x > 0$,

$$\begin{aligned} & E\left[\int_0^T (\Gamma_{0,s}^{\hat{\beta}, \hat{\gamma}} F(s, \hat{\beta}_s, \hat{\gamma}_s) + \hat{\zeta} N_{0,s}^{\hat{\mu}, \hat{\nu}} \tilde{b}(s, \hat{\mu}_s, \hat{\nu}_s)) ds + \Gamma_{0,T}^{\hat{\beta}, \hat{\gamma}} \tilde{u}\left(\hat{\zeta} \frac{N_{0,T}^{\hat{\mu}, \hat{\nu}}}{\Gamma_{0,T}^{\hat{\beta}, \hat{\gamma}}}\right) + \hat{\zeta} x\right] \\ &= \tilde{V}(\hat{\zeta}) + \hat{\zeta} x \\ &\leq \tilde{V}(\zeta) + \zeta x \\ &\leq E\left[\int_0^T (\Gamma_{0,s}^{\beta, \gamma} F(s, \beta_s, \gamma_s) + \zeta N_{0,s}^{\mu, \nu} \tilde{b}(s, \mu_s, \nu_s)) ds + \Gamma_{0,T}^{\beta, \gamma} \tilde{u}\left(\zeta \frac{N_{0,T}^{\mu, \nu}}{\Gamma_{0,T}^{\beta, \gamma}}\right) + \zeta x\right]. \end{aligned}$$

This completes the proof. \square

Assumption 3.4 *There exist some nonnegative numbers k_3, k_4, p_1 such that*

$$|I(\zeta)| \leq k_3 + k_4 \zeta^{p_1}, \quad \forall \zeta > 0.$$

Our main result is the following theorem.

Theorem 3.5 Under Assumption 2.4, 2.5, 2.6, 2.9 and 3.4, let $(\hat{\zeta}, \hat{\mu}, \hat{\nu}, \hat{\beta}, \hat{\gamma})$ as in lemma 3.3 and define

$$\hat{\xi} = I\left(\hat{\zeta} \frac{N_{0,T}^{\hat{\mu}, \hat{\nu}}}{\Gamma_{0,T}^{\hat{\beta}, \hat{\gamma}}}\right) \text{ a.s.}$$

Then we have $\forall \xi \in \mathcal{A}(x)$, $\forall (\beta, \gamma) \in \mathcal{B}$,

$$\begin{aligned} & E\left[\int_0^T \Gamma_{0,s}^{\hat{\beta}, \hat{\gamma}} F(s, \hat{\beta}_s, \hat{\gamma}_s) ds + \Gamma_{0,T}^{\hat{\beta}, \hat{\gamma}} u(\xi)\right] \\ & \leq E\left[\int_0^T \Gamma_{0,s}^{\hat{\beta}, \hat{\gamma}} F(s, \hat{\beta}_s, \hat{\gamma}_s) ds + \Gamma_{0,T}^{\hat{\beta}, \hat{\gamma}} u(\hat{\xi})\right] \\ & \leq E\left[\int_0^T \Gamma_{0,s}^{\beta, \gamma} F(s, \beta_s, \gamma_s) ds + \Gamma_{0,T}^{\beta, \gamma} u(\hat{\xi})\right]. \end{aligned} \quad (3.6)$$

That is to say, $(\hat{\xi}, \hat{\beta}, \hat{\gamma})$ is a saddle point of problem (2.18).

Proof: The proof is divided into three steps.

Step 1. We prove that $\forall (\beta, \gamma) \in \mathcal{B}$,

$$E\left[\int_0^T \Gamma_{0,s}^{\hat{\beta}, \hat{\gamma}} F(s, \hat{\beta}_s, \hat{\gamma}_s) ds + \Gamma_{0,T}^{\hat{\beta}, \hat{\gamma}} u(\hat{\xi})\right] \leq E\left[\int_0^T \Gamma_{0,s}^{\beta, \gamma} F(s, \beta_s, \gamma_s) ds + \Gamma_{0,T}^{\beta, \gamma} u(\hat{\xi})\right].$$

By the boundedness of $\hat{\mu}, \hat{\nu}, \hat{\beta}, \hat{\gamma}$, we have $\frac{N_{0,T}^{\hat{\mu}, \hat{\nu}}}{\Gamma_{0,T}^{\hat{\beta}, \hat{\gamma}}} \in L^p$ for any $p \geq 1$. Then, Assumption 3.4 guarantees $\hat{\xi} \in L^2$. By lemma 3.1,

$$\begin{aligned} \tilde{V}(\hat{\zeta}) &= \inf_{\substack{(\beta, \gamma) \in \mathcal{B} \\ (\mu, \nu) \in \mathcal{B}'}} E\left[\int_0^T (\Gamma_{0,s}^{\beta, \gamma} F(s, \beta_s, \gamma_s) + \hat{\zeta} N_{0,s}^{\mu, \nu} \tilde{b}(s, \mu_s, \nu_s)) ds + \Gamma_{0,T}^{\beta, \gamma} \tilde{u}\left(\hat{\zeta} \frac{N_{0,T}^{\mu, \nu}}{\Gamma_{0,T}^{\beta, \gamma}}\right)\right] \\ &= E\left[\int_0^T (\Gamma_{0,s}^{\hat{\beta}, \hat{\gamma}} F(s, \hat{\beta}_s, \hat{\gamma}_s) + \hat{\zeta} N_{0,s}^{\hat{\mu}, \hat{\nu}} \tilde{b}(s, \hat{\mu}_s, \hat{\nu}_s)) ds + \Gamma_{0,T}^{\hat{\beta}, \hat{\gamma}} \tilde{u}\left(\hat{\zeta} \frac{N_{0,T}^{\hat{\mu}, \hat{\nu}}}{\Gamma_{0,T}^{\hat{\beta}, \hat{\gamma}}}\right)\right]. \end{aligned}$$

It yields that $(\hat{\beta}, \hat{\gamma})$ is an optimal control of the following optimization problem:

$$\inf_{(\beta, \gamma) \in \mathcal{B}} E\left[\int_0^T (\Gamma_{0,s}^{\beta, \gamma} F(s, \beta_s, \gamma_s) ds + \Gamma_{0,T}^{\beta, \gamma} \tilde{u}\left(\hat{\zeta} \frac{N_{0,T}^{\hat{\mu}, \hat{\nu}}}{\Gamma_{0,T}^{\beta, \gamma}}\right))\right] \quad (3.7)$$

subject to

$$\begin{cases} d\Gamma_{0,t}^{\beta, \gamma} = \Gamma_{0,t}^{\beta, \gamma} \beta_t dt + \Gamma_{0,t}^{\beta, \gamma} \gamma_t' dW_t, \\ \Gamma_{0,0}^{\beta, \gamma} = 1, \end{cases}$$

where $\Gamma_{0,t}^{\beta, \gamma}$ is the state process at time t . Applying the maximum principle in Peng [26], we obtain a necessary condition for $(\hat{\beta}, \hat{\gamma})$:

$$F(t, \beta_t, \gamma_t) + p_t \beta_t + q_t \gamma_t \geq F(t, \hat{\beta}_t, \hat{\gamma}_t) + p_t \hat{\beta}_t + q_t \hat{\gamma}_t, \quad \forall (\beta, \gamma) \in \mathcal{B}, \quad (3.8)$$

where (p_t, q_t) is the solution of the following adjoint equation

$$\begin{cases} -dp_t = (F(t, \hat{\beta}_t, \hat{\gamma}_t) + p_t \hat{\beta}_t + q_t' \hat{\gamma}_t) dt - q_t' dW_t, \\ p_T = u\left(I\left(\hat{\zeta} \frac{N_{0,T}^{\hat{\mu}, \hat{\nu}}}{\Gamma_{0,T}^{\hat{\beta}, \hat{\gamma}}}\right)\right). \end{cases} \quad (3.9)$$

$\forall(\beta, \gamma) \in \mathcal{B}$, let (y_t, z_t) and $(\tilde{y}_t, \tilde{z}_t)$ be the unique solutions for the following two linear BSDEs respectively,

$$y_t = u(\hat{\xi}) + \int_t^T (y_s \hat{\beta}_s + z'_s \hat{\gamma}_s + F(s, \hat{\beta}_s, \hat{\gamma}_s)) ds - \int_t^T z'_s dW_s, \quad (3.10)$$

$$\tilde{y}_t = u(\hat{\xi}) + \int_t^T (\tilde{y}_s \beta_s + \tilde{z}'_s \gamma_s + F(s, \beta_s, \gamma_s)) ds - \int_t^T \tilde{z}'_s dW_s. \quad (3.11)$$

Note that (3.9) and (3.10) share the same solution. Thus, (3.8) can be written as

$$F(t, \beta_t, \gamma_t) + y_t \beta_t + z_t \gamma_t \geq F(t, \hat{\beta}_t, \hat{\gamma}_t) + y_t \hat{\beta}_t + z_t \hat{\gamma}_t, \quad \forall(\beta, \gamma) \in \mathcal{B}. \quad (3.12)$$

By (3.12) and the comparison theorem of BSDE, $y_t \leq \tilde{y}_t$, a.s.. Especially, we have $y_0 \leq \tilde{y}_0$. It is easy to see that

$$y_0 = E\left[\int_0^T \Gamma_{0,s}^{\hat{\beta}, \hat{\gamma}} F(s, \hat{\beta}_s, \hat{\gamma}_s) ds + \Gamma_{0,T}^{\hat{\beta}, \hat{\gamma}} u(\hat{\xi})\right],$$

and

$$\tilde{y}_0 = E\left[\int_0^T \Gamma_{0,s}^{\beta, \gamma} F(s, \beta_s, \gamma_s) ds + \Gamma_{0,T}^{\beta, \gamma} u(\hat{\xi})\right].$$

Thus,

$$E\left[\int_0^T \Gamma_{0,s}^{\hat{\beta}, \hat{\gamma}} F(s, \hat{\beta}_s, \hat{\gamma}_s) ds + \Gamma_{0,T}^{\hat{\beta}, \hat{\gamma}} u(\hat{\xi})\right] \leq E\left[\int_0^T \Gamma_{0,s}^{\beta, \gamma} F(s, \beta_s, \gamma_s) ds + \Gamma_{0,T}^{\beta, \gamma} u(\hat{\xi})\right], \quad \forall(\beta, \gamma) \in \mathcal{B}.$$

Step 2. We prove that

$$E[\hat{\xi} N_{0,T}^{\hat{\mu}, \hat{\nu}}] = x + E \int_0^T N_{0,s}^{\hat{\mu}, \hat{\nu}} \tilde{b}(s, \hat{\mu}_s, \hat{\nu}_s) ds. \quad (3.13)$$

From the convexity of \tilde{u} and Lemma 3.1, we get $\tilde{V}(\cdot)$ is convex. Fix $\zeta > 0$. Then for any $\delta > 0$, we have

$$\begin{aligned} \frac{\tilde{V}(\zeta + \delta) - \tilde{V}(\zeta)}{\delta} &\leq \frac{E\left[\Gamma_{0,T}^{\hat{\beta}, \hat{\gamma}} \tilde{u}\left((\zeta + \delta) \frac{N_{0,T}^{\hat{\mu}, \hat{\nu}}}{\Gamma_{0,T}^{\hat{\beta}, \hat{\gamma}}}\right) - \Gamma_{0,T}^{\hat{\beta}, \hat{\gamma}} \tilde{u}\left(\zeta \frac{N_{0,T}^{\hat{\mu}, \hat{\nu}}}{\Gamma_{0,T}^{\hat{\beta}, \hat{\gamma}}}\right)\right]}{\delta} + E \int_0^T N_{0,s}^{\hat{\mu}, \hat{\nu}} \tilde{b}(s, \hat{\mu}_s, \hat{\nu}_s) ds \\ &\leq E\left[N_{0,T}^{\hat{\mu}, \hat{\nu}} \tilde{u}'\left((\zeta + \delta) \frac{N_{0,T}^{\hat{\mu}, \hat{\nu}}}{\Gamma_{0,T}^{\hat{\beta}, \hat{\gamma}}}\right)\right] + E \int_0^T N_{0,s}^{\hat{\mu}, \hat{\nu}} \tilde{b}(s, \hat{\mu}_s, \hat{\nu}_s) ds \\ &= -E\left[N_{0,T}^{\hat{\mu}, \hat{\nu}} I\left((\zeta + \delta) \frac{N_{0,T}^{\hat{\mu}, \hat{\nu}}}{\Gamma_{0,T}^{\hat{\beta}, \hat{\gamma}}}\right)\right] + E \int_0^T N_{0,s}^{\hat{\mu}, \hat{\nu}} \tilde{b}(s, \hat{\mu}_s, \hat{\nu}_s) ds. \end{aligned}$$

By Levi's lemma,

$$\lim_{\delta \rightarrow 0+} \frac{\tilde{V}(\zeta + \delta) - \tilde{V}(\zeta)}{\delta} \leq -E\left[N_{0,T}^{\hat{\mu}, \hat{\nu}} I\left(\zeta \frac{N_{0,T}^{\hat{\mu}, \hat{\nu}}}{\Gamma_{0,T}^{\hat{\beta}, \hat{\gamma}}}\right)\right] + E \int_0^T N_{0,s}^{\hat{\mu}, \hat{\nu}} \tilde{b}(s, \hat{\mu}_s, \hat{\nu}_s) ds. \quad (3.14)$$

We can similarly deduce that

$$\lim_{\delta \rightarrow 0+} \frac{\tilde{V}(\zeta) - \tilde{V}(\zeta - \delta)}{\delta} \geq -E\left[N_{0,T}^{\hat{\mu}, \hat{\nu}} I\left(\zeta \frac{N_{0,T}^{\hat{\mu}, \hat{\nu}}}{\Gamma_{0,T}^{\hat{\beta}, \hat{\gamma}}}\right)\right] + E \int_0^T N_{0,s}^{\hat{\mu}, \hat{\nu}} \tilde{b}(s, \hat{\mu}_s, \hat{\nu}_s) ds. \quad (3.15)$$

So $\tilde{V}(\cdot)$ is differentiable on $(0, \infty)$ and

$$\tilde{V}'(\zeta) = -E\left[N_{0,T}^{\hat{\mu}, \hat{\nu}} I\left(\zeta \frac{N_{0,T}^{\hat{\mu}, \hat{\nu}}}{\Gamma_{0,T}^{\hat{\beta}, \hat{\gamma}}}\right)\right] + E \int_0^T N_{0,s}^{\hat{\mu}, \hat{\nu}} \tilde{b}(s, \hat{\mu}_s, \hat{\nu}_s) ds. \quad (3.16)$$

From lemma 3.2, we know $\hat{\zeta} \in (0, \infty)$ attains the infimum of $\inf_{\zeta > 0} (\tilde{V}(\zeta) + \zeta x)$. Then $\tilde{V}'(\hat{\zeta}) = -x$. Combined with (3.16), we derive (3.13).

Step 3. We prove that $\forall \xi \in \mathcal{A}(x)$,

$$E\left[\int_0^T \Gamma_{0,s}^{\hat{\beta}, \hat{\gamma}} F(s, \hat{\beta}_s, \hat{\gamma}_s) ds + \Gamma_{0,T}^{\hat{\beta}, \hat{\gamma}} u(\xi)\right] \leq E\left[\int_0^T \Gamma_{0,s}^{\hat{\beta}, \hat{\gamma}} F(s, \hat{\beta}_s, \hat{\gamma}_s) ds + \Gamma_{0,T}^{\hat{\beta}, \hat{\gamma}} u(\hat{\xi})\right].$$

By the definition of $\tilde{u}(\cdot)$ and (2.17), $\forall \xi \in \mathcal{A}(x)$,

$$\begin{aligned} & E\left[\int_0^T \Gamma_{0,s}^{\hat{\beta}, \hat{\gamma}} F(s, \hat{\beta}_s, \hat{\gamma}_s) ds + \Gamma_{0,T}^{\hat{\beta}, \hat{\gamma}} u(\xi)\right] \\ & \leq E\left[\int_0^T \Gamma_{0,s}^{\hat{\beta}, \hat{\gamma}} F(s, \hat{\beta}_s, \hat{\gamma}_s) ds + \Gamma_{0,T}^{\hat{\beta}, \hat{\gamma}} \tilde{u}\left(\zeta \frac{N_{0,T}^{\hat{\mu}, \hat{\nu}}}{\Gamma_{0,T}^{\hat{\beta}, \hat{\gamma}}}\right) + \zeta \xi N_{0,T}^{\hat{\mu}, \hat{\nu}}\right] \\ & \leq E\left[\int_0^T (\Gamma_{0,s}^{\hat{\beta}, \hat{\gamma}} F(s, \hat{\beta}_s, \hat{\gamma}_s) + \hat{\zeta} N_{0,s}^{\hat{\mu}, \hat{\nu}} \tilde{b}(s, \hat{\mu}_s, \hat{\nu}_s)) ds + \Gamma_{0,T}^{\hat{\beta}, \hat{\gamma}} \tilde{u}\left(\hat{\zeta} \frac{N_{0,T}^{\hat{\mu}, \hat{\nu}}}{\Gamma_{0,T}^{\hat{\beta}, \hat{\gamma}}}\right)\right] + \hat{\zeta} x \\ & = E\left[\int_0^T \Gamma_{0,s}^{\hat{\beta}, \hat{\gamma}} F(s, \hat{\beta}_s, \hat{\gamma}_s) ds + \Gamma_{0,T}^{\hat{\beta}, \hat{\gamma}} u(\hat{\xi})\right]. \end{aligned}$$

The last equality is due to (3.13).

This completes the proof. \square

4 Investors with ambiguity aversion

For any d -dimensional vector $K = (K_1, \dots, K_d)'$, denote by $|K|$ the d -dimensional vector with i th component $|K_i|$, $i = 1, \dots, d$; denote by $\|K\|$ the Euclidean norm $(\sum_{i=1}^d K_i^2)^{\frac{1}{2}}$. We say that two d -dimensional vectors $K \geq \tilde{K}$ if $K_i \geq \tilde{K}_i$, $i = 1, \dots, d$. For any $K, \tilde{K}, \hat{K} \in \mathbb{R}^d$, denote by $K \vee \tilde{K} \wedge \hat{K}$ the d -dimensional vector with i th component $K_i \vee \tilde{K}_i \wedge \hat{K}_i$.

We model the utility process (2.7) via

$$f(t, y, z) = -K'|z|, \quad (4.1)$$

where K is a given d -dimensional vector whose component $K_i \geq 0$, $i = 1, 2, \dots, d$. Chen and Epstein [2] interpreted the term $-K'|z|$ as modeling ambiguity aversion rather than risk aversion. This special formulation in [2] is called K-ignorance. When $K_i = 0$, $i = 1, 2, \dots, d$, it degenerates to the classical expected utility maximization problem studied in [3], [19] etc.

For the K-ignorance case, we have

$$F(\omega, t, \beta, \gamma) \equiv 0$$

and

$$\mathcal{B} = \{(\beta, \gamma) \mid \beta_t = 0, \gamma'_t = (\gamma_{1t}, \dots, \gamma_{dt}) \text{ is } \{\mathcal{F}_t\}_{t \geq 0} \text{-progressively measurable and } |\gamma_t| \leq K, t \in [0, T], a.s.\}.$$

Define

$$\mathcal{B}_2 = \{\gamma \mid \gamma'_t = (\gamma_{1t}, \dots, \gamma_{dt}) \text{ is } \{\mathcal{F}_t\}_{t \geq 0} \text{-progressively measurable and } |\gamma_t| \leq K, t \in [0, T], a.s.\}.$$

For a given $\gamma \in \mathcal{B}_2$, define

$$\mathcal{B}_{2,t}(\gamma) = \{\tilde{\gamma} \in \mathcal{B}_2 \mid \tilde{\gamma}_s \equiv \gamma_s \text{ on } [0, t], \text{ a.s.}\}.$$

For some $0 < \alpha < 1$, set

$$u(x) = \frac{1}{\alpha} x^\alpha, \quad x \geq 0.$$

It is easy to check that u satisfies Assumption 2.6, 3.4 and for any $\zeta > 0$,

$$\begin{aligned} I(\zeta) &= \zeta^{\frac{1}{\alpha-1}}, \\ \tilde{u}(\zeta) &= u(I(\zeta)) - \zeta I(\zeta) = \frac{1-\alpha}{\alpha} \zeta^{\frac{\alpha}{\alpha-1}}. \end{aligned}$$

In this section, we assume that the investors have the same recursive utility as above. In the following, we investigate three different kinds of wealth equations.

4.1 Linear wealth equation

In Example 2.1, suppose that $r_t \equiv 0$ and $\sigma_t \equiv I_{d \times d}$. Then, the wealth equation becomes

$$\begin{cases} dX_t = \pi'_t b_t dt + \pi_t dW_t, \\ X_0 = x. \end{cases}$$

where b_t is a uniformly bounded progressively measurable process.

In this case, $\tilde{b} \equiv 0$ and

$$\mathcal{B}' = \{(\mu, \nu) \mid \mu_t = 0, \nu_t = b_t, t \in [0, T], \text{ a.s.}\}.$$

Then the value function $\tilde{V}(\zeta)$ in (3.2) becomes

$$\tilde{V}(\zeta) = \inf_{\gamma \in \mathcal{B}_2} E\left[\Gamma_{0,T}^{0,\gamma} \tilde{u}\left(\zeta \frac{N_{0,T}^{0,b}}{\Gamma_{0,T}^{0,\gamma}}\right)\right] = \frac{1-\alpha}{\alpha} \zeta^{\frac{\alpha}{\alpha-1}} \inf_{\gamma \in \mathcal{B}_2} E\left[(N_{0,T}^{0,b})^{\frac{\alpha}{\alpha-1}} (\Gamma_{0,T}^{0,\gamma})^{\frac{1}{1-\alpha}}\right]. \quad (4.2)$$

Define

$$\tilde{V}(t, \gamma, \zeta) = \frac{1-\alpha}{\alpha} \zeta^{\frac{\alpha}{\alpha-1}} \operatorname{ess\,inf}_{\tilde{\gamma} \in \mathcal{B}_{2,t}(\gamma)} E\left[(N_{0,T}^{0,b})^{\frac{\alpha}{\alpha-1}} (\Gamma_{0,T}^{0,\tilde{\gamma}})^{\frac{1}{1-\alpha}} \mid \mathcal{F}_t\right].$$

We conjecture that $\tilde{V}(t, \gamma, \zeta)$ has the following form

$$\tilde{V}(t, \gamma, \zeta) = \frac{1-\alpha}{\alpha} \zeta^{\frac{\alpha}{\alpha-1}} (N_{0,t}^{0,b})^{\frac{\alpha}{\alpha-1}} (\Gamma_{0,t}^{0,\gamma})^{\frac{1}{1-\alpha}} e^{\tilde{Y}_t},$$

where (\tilde{Y}, \tilde{Z}) is the solution of the following BSDE

$$\tilde{Y}_t = \int_t^T g(s, \tilde{Z}_s) ds - \int_t^T \tilde{Z}_s' dW_s. \quad (4.3)$$

The generator $g(t, z)$ of (4.3) can be determined via the following martingale principle in [7]. The readers may also refer to Hu et al [13].

Lemma 4.1 *The process $\tilde{V}(t, \gamma, \zeta)$ is a submartingale for any $\gamma \in \mathcal{B}_2$ and $\hat{\gamma}$ is the solution of problem (4.2) if and only if $\tilde{V}(t, \hat{\gamma}, \zeta)$ is a martingale.*

Applying Itô's formula to $(N_{0,t}^{0,b})^{\frac{\alpha}{\alpha-1}} (\Gamma_{0,t}^{0,\gamma})^{\frac{1}{1-\alpha}} e^{\tilde{Y}_t}$,

$$\begin{aligned} & d(N_{0,t}^{0,b})^{\frac{\alpha}{\alpha-1}} (\Gamma_{0,t}^{0,\gamma})^{\frac{1}{1-\alpha}} e^{\tilde{Y}_t} \\ &= (N_{0,t}^{0,b})^{\frac{\alpha}{\alpha-1}} (\Gamma_{0,t}^{0,\gamma})^{\frac{1}{1-\alpha}} e^{\tilde{Y}_t} \left[-g(t, \tilde{Z}_t) + \frac{1}{2} \|\tilde{Z}_t\|^2 + \frac{1}{2} \frac{\alpha}{(1-\alpha)^2} \|b_t + \gamma_t\|^2 + \frac{1}{1-\alpha} \tilde{Z}_t' \gamma_t + \frac{\alpha}{1-\alpha} \tilde{Z}_t' b_t \right] dt \\ & \quad + (N_{0,t}^{0,b})^{\frac{\alpha}{\alpha-1}} (\Gamma_{0,t}^{0,\gamma})^{\frac{1}{1-\alpha}} e^{\tilde{Y}_t} \left[\frac{1}{1-\alpha} \gamma_t' + \frac{\alpha}{1-\alpha} b_t' + \tilde{Z}_t' \right] dW_t. \end{aligned}$$

According to Lemma 4.1, we have

$$-g(t, \tilde{Z}_t) + \frac{1}{2} \|\tilde{Z}_t\|^2 + \frac{1}{2} \frac{\alpha}{(1-\alpha)^2} \|b_t + \gamma_t\|^2 + \frac{1}{1-\alpha} \gamma_t' \tilde{Z}_t + \frac{\alpha}{1-\alpha} b_t' \tilde{Z}_t \geq 0, \quad \forall \gamma \in \mathcal{B}_2.$$

For $z \in \mathbb{R}^d$ and $t \in [0, T]$, define

$$\begin{aligned} g(t, z) &= \operatorname{essinf}_{\gamma \in \mathcal{B}_2} \left[\frac{1}{2} \frac{\alpha}{(1-\alpha)^2} \|b_t + \gamma_t\|^2 + \frac{1}{1-\alpha} \gamma_t' z + \frac{\alpha}{1-\alpha} b_t' z + \frac{1}{2} \|z\|^2 \right] \\ &= \frac{1}{2} \frac{\alpha}{(1-\alpha)^2} \operatorname{essinf}_{\gamma \in \mathcal{B}_2} \left[\|\gamma_t + b_t + \frac{1-\alpha}{\alpha} z\|^2 - 2 \frac{(1-\alpha)^2}{\alpha} b_t' z - \frac{(1-\alpha)^3}{\alpha^2} \|z\|^2 \right] \\ &= \frac{1}{2} \frac{\alpha}{(1-\alpha)^2} \left[\operatorname{dist}_{\mathcal{B}_2}^2(b_t + \frac{1-\alpha}{\alpha} z) - 2 \frac{(1-\alpha)^2}{\alpha} b_t' z - \frac{(1-\alpha)^3}{\alpha^2} \|z\|^2 \right], \end{aligned}$$

where $\operatorname{dist}_{\mathcal{B}_2}^2(b_t + \frac{1-\alpha}{\alpha} z) = \operatorname{essinf}_{\gamma \in \mathcal{B}_2} \|\gamma_t + b_t + \frac{1-\alpha}{\alpha} z\|^2$. By the result of Kobylanski [22], the quadratic BSDE (4.3) has a unique solution (\tilde{Y}, \tilde{Z}) .

Then the infimum in (4.2) is attained at

$$\hat{\gamma}_t = (-K) \vee \left(-b_t - \frac{1-\alpha}{\alpha} \tilde{Z}_t \right) \wedge K, \quad t \in [0, T], \quad a.s.$$

and

$$\tilde{V}(\zeta) = \frac{1-\alpha}{\alpha} \zeta^{\frac{\alpha}{\alpha-1}} E[(N_{0,T}^{0,b})^{\frac{\alpha}{\alpha-1}} (\Gamma_{0,T}^{0,\hat{\gamma}})^{\frac{1}{1-\alpha}}], \quad \zeta > 0.$$

The second infimum in (3.3) is attained at

$$\hat{\zeta} = x^{\alpha-1} \left(E[(N_{0,T}^{0,b})^{\frac{\alpha}{\alpha-1}} (\Gamma_{0,T}^{0,\hat{\gamma}})^{\frac{1}{1-\alpha}}] \right)^{1-\alpha}.$$

Thus, the optimal terminal wealth is given by

$$\hat{\xi} = I(\hat{\zeta} \frac{N_{0,T}^{0,b}}{\Gamma_{0,T}^{0,\hat{\gamma}}}) = x \left(E[(N_{0,T}^{0,b})^{\frac{\alpha}{\alpha-1}} (\Gamma_{0,T}^{0,\hat{\gamma}})^{\frac{1}{1-\alpha}}] \right)^{-1} (N_{0,T}^{0,b})^{\frac{1}{\alpha-1}} (\Gamma_{0,T}^{0,\hat{\gamma}})^{\frac{1}{1-\alpha}}.$$

It is easy to check the following propositions.

Proposition 4.2 *When b_t is a deterministic function, we have $\tilde{Z}_t = 0$ and*

$$\hat{\gamma}_t = (-K) \vee (-b_t) \wedge K, \quad t \in [0, T].$$

4.2 Higher interest rate for borrowing

In this subsection, for simplicity, we assume that all variables are 1-dimensional.

In Example 2.2, the investor is allowed to borrow money with interest rate $R_t \geq r_t$, $t \in [0, T]$. Suppose that b , R , r are deterministic continuous functions and $\sigma_t \equiv 1$. Then the wealth equation becomes

$$\begin{cases} dX_t = (r_t X_t + \pi_t(b_t - r_t) - (R_t - r_t)(X_t - \pi_t)^-) dt + \pi_t dW_t, \\ X_0 = x. \end{cases}$$

In this case, $\tilde{b} \equiv 0$,

$$\mathcal{B}' = \{(\mu, \nu) \mid (\mu_t, \nu_t) \text{ is } \{\mathcal{F}_t\}_{t \geq 0}\text{-progressively measurable and } r_t \leq \mu_t \leq R_t, \mu_t + \nu_t = b_t, t \in [0, T], a.s.\}.$$

and

$$\mathcal{B}'_2 = \{\mu \mid \mu_t \text{ is } \{\mathcal{F}_t\}_{t \geq 0}\text{-progressively measurable and } r_t \leq \mu_t \leq R_t, t \in [0, T], a.s.\}.$$

The value function $\tilde{V}(\zeta)$ in (3.2) becomes

$$\begin{aligned} \tilde{V}(\zeta) &= \inf_{\substack{(\beta, \gamma) \in \mathcal{B} \\ (\mu, \nu) \in \mathcal{B}'}} E[\Gamma_{0,T}^{\beta, \gamma} \tilde{u}(\zeta \frac{N_{0,T}^{\mu, \nu}}{\Gamma_{0,T}^{\beta, \gamma}})] \\ &= \inf_{\mu \in \mathcal{B}'_2, \gamma \in \mathcal{B}_2} E[\Gamma_{0,T}^{0, \gamma} \tilde{u}(\zeta \frac{N_{0,T}^{\mu, b-\mu}}{\Gamma_{0,T}^{0, \gamma}})] \\ &= \frac{1-\alpha}{\alpha} \zeta^{\frac{\alpha}{\alpha-1}} \inf_{\mu \in \mathcal{B}'_2, \gamma \in \mathcal{B}_2} E[(N_{0,T}^{\mu, b-\mu})^{\frac{\alpha}{\alpha-1}} (\Gamma_{0,T}^{0, \gamma})^{\frac{1}{1-\alpha}}]. \end{aligned} \quad (4.4)$$

Define

$$\tilde{V}(t, \mu, \gamma, \zeta) = \frac{1-\alpha}{\alpha} \zeta^{\frac{\alpha}{\alpha-1}} \operatorname{ess\,inf}_{\tilde{\mu} \in \mathcal{B}'_{2,t}(\mu), \tilde{\gamma} \in \mathcal{B}_{2,t}(\gamma)} E[(N_{0,T}^{\mu, b-\tilde{\mu}})^{\frac{\alpha}{\alpha-1}} (\Gamma_{0,T}^{0, \tilde{\gamma}})^{\frac{1}{1-\alpha}} \mid \mathcal{F}_t],$$

where $\mathcal{B}'_{2,t}(\mu) = \{\tilde{\mu} \in \mathcal{B}'_2 \mid \tilde{\mu}_s = \mu_s \text{ on } [0, t]\}$.

We conjecture that $\tilde{V}(t, \mu, \gamma, \zeta)$ has the following form:

$$\tilde{V}(t, \mu, \gamma, \zeta) = \frac{1-\alpha}{\alpha} \zeta^{\frac{\alpha}{\alpha-1}} (N_{0,t}^{\mu, b-\mu})^{\frac{\alpha}{\alpha-1}} (\Gamma_{0,t}^{0, \gamma})^{\frac{1}{1-\alpha}} e^{\tilde{Y}_t},$$

where (\tilde{Y}, \tilde{Z}) is the solution of the BSDE

$$\tilde{Y}_t = \int_t^T g(s, \tilde{Z}_s) ds - \int_t^T \tilde{Z}_s dW_s. \quad (4.5)$$

Similarly, $\forall \mu \in \mathcal{B}'_2$ and $\gamma \in \mathcal{B}_2$, the process $\tilde{V}(t, \mu, \gamma, \zeta)$ is a submartingale. $\hat{\mu}$ and $\hat{\gamma}$ are the optimal solutions of problem (4.4) if and only if $\tilde{V}(t, \hat{\mu}, \hat{\gamma}, \zeta)$ is a martingale. Now we determine $g(t, z)$ via this martingale principle.

Applying Itô's formula to $(N_{0,t}^{\mu, b-\mu})^{\frac{\alpha}{\alpha-1}} (\Gamma_{0,t}^{0, \gamma})^{\frac{1}{1-\alpha}} e^{\tilde{Y}_t}$,

$$\begin{aligned} & d(N_{0,t}^{\mu, b-\mu})^{\frac{\alpha}{\alpha-1}} (\Gamma_{0,t}^{0, \gamma})^{\frac{1}{1-\alpha}} e^{\tilde{Y}_t} \\ &= (N_{0,t}^{\mu, b-\mu})^{\frac{\alpha}{\alpha-1}} (\Gamma_{0,t}^{0, \gamma})^{\frac{1}{1-\alpha}} e^{\tilde{Y}_t} \left[-g(t, \tilde{Z}_t) + \frac{1}{2} \tilde{Z}_t^2 + \frac{1}{2} \frac{\alpha}{(1-\alpha)^2} (b_t + \gamma_t - \mu_t)^2 + \frac{\alpha}{1-\alpha} \mu_t \right. \\ & \quad \left. + \frac{1}{1-\alpha} \tilde{Z}_t (\gamma_t + \alpha(b_t - \mu_t)) \right] dt + (N_{0,t}^{\mu, b-\mu})^{\frac{\alpha}{\alpha-1}} (\Gamma_{0,t}^{0, \gamma})^{\frac{1}{1-\alpha}} e^{\tilde{Y}_t} \left[\frac{1}{1-\alpha} \gamma_t + \frac{\alpha}{1-\alpha} (b_t - \mu_t) + \tilde{Z}_t \right] dW_t. \end{aligned}$$

Define

$$g(t, z) = \operatorname{ess\,inf}_{\mu \in \mathcal{B}'_2, \gamma \in \mathcal{B}_2} \left[\frac{1}{2} \frac{\alpha}{(1-\alpha)^2} (b_t + \gamma_t - \mu_t)^2 + \frac{\alpha}{1-\alpha} \mu_t + \frac{1}{1-\alpha} z(\gamma_t + \alpha(b_t - \mu_t)) + \frac{1}{2} z^2 \right].$$

Since b_t is a deterministic function, $g(t, z)$ is also deterministic. By the existence and uniqueness theorem of BSDE, the solution of (4.5) satisfies $\tilde{Z}_t = 0$, $t \in [0, T]$. Thus the optimal solutions $\hat{\mu}$ and $\hat{\gamma}$ attain the infimum in

$$g(t, 0) = \frac{1}{2} \frac{\alpha}{(1-\alpha)^2} \inf_{\mu \in \mathcal{B}'_2, \gamma \in \mathcal{B}_2} [(b_t + \gamma_t - \mu_t)^2 + 2(1-\alpha)\mu_t]. \quad (4.6)$$

Set

$$H(\mu, \gamma) = (b_t + \gamma - \mu)^2 + 2(1-\alpha)\mu.$$

It is easy to check that the following equations

$$\begin{cases} \frac{\partial H}{\partial \mu} = -2(b_t + \gamma - \mu) + 2(1-\alpha) = 0, \\ \frac{\partial H}{\partial \gamma} = 2(b_t + \gamma - \mu) = 0 \end{cases}$$

have no solutions. So the infimum in (4.6) must be attained at the boundary of the region $[r_t, R_t] \times [-K, K]$.

For the optimal solutions $\hat{\mu}$ and $\hat{\gamma}$, we have

$$\tilde{V}(\zeta) = \frac{1-\alpha}{\alpha} \zeta^{\frac{\alpha}{\alpha-1}} E[(N_{0,T}^{\hat{\mu}, b-\hat{\mu}})^{\frac{\alpha}{\alpha-1}} (\Gamma_{0,T}^{0, \hat{\gamma}})^{\frac{1}{1-\alpha}}], \quad \zeta > 0.$$

The second infimum in (3.3) is attained at

$$\hat{\zeta} = x^{\alpha-1} (E[(N_{0,T}^{\hat{\mu}, b-\hat{\mu}})^{\frac{\alpha}{\alpha-1}} (\Gamma_{0,T}^{0, \hat{\gamma}})^{\frac{1}{1-\alpha}}])^{1-\alpha}.$$

Thus, the optimal terminal wealth is given by

$$\hat{\xi} = I(\hat{\zeta} \frac{N_{0,T}^{\hat{\mu}, b-\hat{\mu}}}{\Gamma_{0,T}^{0, \hat{\gamma}}}) = x(E[(N_{0,T}^{\hat{\mu}, b-\hat{\mu}})^{\frac{\alpha}{\alpha-1}} (\Gamma_{0,T}^{0, \hat{\gamma}})^{\frac{1}{1-\alpha}}])^{-1} (N_{0,T}^{\hat{\mu}, b-\hat{\mu}})^{\frac{1}{\alpha-1}} (\Gamma_{0,T}^{0, \hat{\gamma}})^{\frac{1}{1-\alpha}}.$$

We can easily deduce the following propositions.

Proposition 4.3 *When $K \equiv 0$, we obtain $\hat{\gamma} \equiv 0$ and $\hat{\mu}_t = r_t \vee (b_t - 1 + \alpha) \wedge R_t$. This coincides with the result of Appendix B in Cvitanic and Karatzas [4].*

Proposition 4.4 *If $\frac{1-\alpha}{2} \leq K \leq b_t - R_t$, $t \in [0, T]$, then the infimum in (4.6) is attained at $\hat{\gamma}_t = -K$, and $\hat{\mu}_t = r_t \vee (b_t - K - 1 + \alpha) \wedge R_t$, $t \in [0, T]$.*

Remark 4.5 *When b, R, r are bounded progressively measurable processes, $g(t, z)$ is no longer deterministic. Similar analysis as in Theorem 7 of Hu et al [13], we can prove the quadratic BSDE (4.5) has a unique solution $(\tilde{Y}_t, \tilde{Z}_t)$. Thanks to the boundedness, closeness and convexity of \mathcal{B}_2 and \mathcal{B}'_2 , there exists a pair $(\hat{\mu}, \hat{\gamma})$ which attains the infimum of $g(t, \tilde{Z}_t)$.*

4.3 Large investor

Suppose that $r \equiv 0$, $\sigma \equiv I_{d \times d}$ and b_t is a deterministic continuous bounded function in Example 2.3. Then, the wealth equation becomes

$$\begin{cases} dX_t = (b'_t \pi_t - \varepsilon' |\pi_t|) dt + \pi'_t \sigma_t dW_t, \\ X_0 = x. \end{cases}$$

In this case, $\tilde{b} \equiv 0$,

$\mathcal{B}' = \{(\mu, \nu) \mid \mu_t = 0, \nu_t = b_t - \delta_t, \delta' = (\delta_1, \dots, \delta_d) \text{ is } \{\mathcal{F}_t\}_{t \geq 0} \text{-progressively measurable and } |\delta_t| \leq \varepsilon, t \in [0, T], a.s.\}$.

and

$\mathcal{B}'_2 = \{\delta \mid \delta'_t = (\delta_{1t}, \dots, \delta_{dt}) \text{ is } \{\mathcal{F}_t\}_{t \geq 0} \text{-progressively measurable and } |\delta_t| \leq \varepsilon, t \in [0, T], a.s.\}$.

The value function $\tilde{V}(\zeta)$ in (3.2) becomes

$$\begin{aligned} \tilde{V}(\zeta) &= \inf_{\substack{(\beta, \gamma) \in \mathcal{B} \\ (\mu, \nu) \in \mathcal{B}'}} E[\Gamma_{0,T}^{\beta, \gamma} \tilde{u}(\zeta \frac{N_{0,T}^{\mu, \nu}}{\Gamma_{0,T}^{\beta, \gamma}})] \\ &= \inf_{\delta \in \mathcal{B}'_2, \gamma \in \mathcal{B}_2} E[\Gamma_{0,T}^{0, \gamma} \tilde{u}(\zeta \frac{N_{0,T}^{0, b-\delta}}{\Gamma_{0,T}^{0, \gamma}})] \\ &= \frac{1-\alpha}{\alpha} \zeta^{\frac{\alpha}{\alpha-1}} \inf_{\delta \in \mathcal{B}'_2, \gamma \in \mathcal{B}_2} E[(N_{0,T}^{0, b-\delta})^{\frac{\alpha}{\alpha-1}} (\Gamma_{0,T}^{0, \gamma})^{\frac{1}{1-\alpha}}]. \end{aligned} \quad (4.7)$$

Consider

$$\tilde{V}(t, \delta, \gamma, \zeta) := \frac{1-\alpha}{\alpha} \zeta^{\frac{\alpha}{\alpha-1}} \operatorname{ess\,inf}_{\tilde{\delta} \in \mathcal{B}'_{2,t}(\delta), \tilde{\gamma} \in \mathcal{B}_{2,t}(\gamma)} E[(N_{0,T}^{0, b-\tilde{\delta}})^{\frac{\alpha}{\alpha-1}} (\Gamma_{0,T}^{0, \tilde{\gamma}})^{\frac{1}{1-\alpha}} | \mathcal{F}_t],$$

where $\mathcal{B}'_{2,t}(\delta) = \{\tilde{\delta} \in \mathcal{B}'_2 \mid \tilde{\delta}_s = \delta_s \text{ on } [0, t]\}$.

We conjecture that $\tilde{V}(t, \delta, \gamma, \zeta)$ has the following form:

$$\tilde{V}(t, \delta, \gamma, \zeta) = \frac{1-\alpha}{\alpha} \zeta^{\frac{\alpha}{\alpha-1}} (N_{0,t}^{0, b-\delta})^{\frac{\alpha}{\alpha-1}} (\Gamma_{0,t}^{0, \gamma})^{\frac{1}{1-\alpha}} e^{\tilde{Y}_t},$$

where (\tilde{Y}, \tilde{Z}) is the solution of the BSDE

$$\tilde{Y}_t = \int_t^T g(s, \tilde{Z}_s) ds - \int_t^T \tilde{Z}'_s dW_s. \quad (4.8)$$

Applying Itô's formula to $(N_{0,t}^{0, b-\delta})^{\frac{\alpha}{\alpha-1}} (\Gamma_{0,t}^{0, \gamma})^{\frac{1}{1-\alpha}} e^{\tilde{Y}_t}$,

$$\begin{aligned} & d(N_{0,t}^{0, b-\delta})^{\frac{\alpha}{\alpha-1}} (\Gamma_{0,t}^{0, \gamma})^{\frac{1}{1-\alpha}} e^{\tilde{Y}_t} \\ &= (N_{0,t}^{0, b-\delta})^{\frac{\alpha}{\alpha-1}} (\Gamma_{0,t}^{0, \gamma})^{\frac{1}{1-\alpha}} e^{\tilde{Y}_t} \left[-g(t, \tilde{Z}_t) + \frac{1}{2} \|\tilde{Z}_t\|^2 + \frac{1}{2} \frac{\alpha}{(1-\alpha)^2} \|b_t + \gamma_t - \delta_t\|^2 + \frac{1}{1-\alpha} \tilde{Z}'_t \gamma_t \right. \\ & \quad \left. + \frac{\alpha}{1-\alpha} \tilde{Z}'_t (b_t - \delta_t) \right] dt + (N_{0,t}^{0, b-\delta})^{\frac{\alpha}{\alpha-1}} (\Gamma_{0,t}^{0, \gamma})^{\frac{1}{1-\alpha}} e^{\tilde{Y}_t} \left[\frac{1}{1-\alpha} \gamma'_t + \frac{\alpha}{1-\alpha} (b'_t - \delta'_t) + \tilde{Z}'_t \right] dW_t. \end{aligned}$$

By the martingale principle, $\forall \delta \in \mathcal{B}'_2, \forall \gamma \in \mathcal{B}_2$,

$$-g(t, \tilde{Z}_t) + \frac{1}{2} \|\tilde{Z}_t\|^2 + \frac{1}{2} \frac{\alpha}{(1-\alpha)^2} \|b_t + \gamma_t - \delta_t\|^2 + \frac{1}{1-\alpha} \tilde{Z}'_t \gamma_t + \frac{\alpha}{1-\alpha} \tilde{Z}'_t (b_t - \delta_t) \geq 0.$$

Define

$$g(t, z) = \inf_{\delta \in \mathcal{B}'_2, \gamma \in \mathcal{B}_2} \left[\frac{1}{2} \frac{\alpha}{(1-\alpha)^2} \|b_t + \gamma_t - \delta_t\|^2 + \frac{1}{1-\alpha} z' \gamma_t + \frac{\alpha}{1-\alpha} (b_t - \delta_t)' z + \frac{1}{2} \|z\|^2 \right].$$

Since b_t is a deterministic function, we know that $g(t, z)$ is deterministic and the solution \tilde{Z} of (4.8) equals 0. Note that

$$g(t, 0) = \frac{1}{2} \frac{\alpha}{(1-\alpha)^2} \inf_{\delta \in \mathcal{B}'_2, \gamma \in \mathcal{B}_2} \|b_t + \gamma_t - \delta_t\|^2.$$

Let $(\hat{\delta}, \hat{\gamma})$ be any continuous functions which attain the minimum of $g(t, 0)$. Then, they also attain the infimum in problem (4.7).

We have

$$\tilde{V}(\zeta) = \frac{1-\alpha}{\alpha} \zeta^{\frac{\alpha}{\alpha-1}} E[(N_{0,T}^{0,b-\hat{\delta}})^{\frac{\alpha}{\alpha-1}} (\Gamma_{0,T}^{0,\hat{\gamma}})^{\frac{1}{1-\alpha}}], \quad \zeta > 0.$$

The second infimum in (3.3) is attained at

$$\hat{\zeta} = x^{\alpha-1} (E[(N_{0,T}^{0,b-\hat{\delta}})^{\frac{\alpha}{\alpha-1}} (\Gamma_{0,T}^{0,\hat{\gamma}})^{\frac{1}{1-\alpha}}])^{1-\alpha} = x^{\alpha-1} e^{\frac{\alpha}{2(1-\alpha)} \int_0^T |b_r + \hat{\gamma}_r - \hat{\delta}_r|^2 dr}.$$

Thus, the optimal terminal wealth is given by

$$\begin{aligned} \hat{\xi} &= I(\hat{\zeta} \frac{N_{0,T}^{0,b-\hat{\delta}}}{\Gamma_{0,T}^{0,\hat{\gamma}}}) \\ &= x (E[(N_{0,T}^{0,b-\hat{\delta}})^{\frac{\alpha}{\alpha-1}} (\Gamma_{0,T}^{0,\hat{\gamma}})^{\frac{1}{1-\alpha}}])^{-1} (N_{0,T}^{0,b-\hat{\delta}})^{\frac{1}{\alpha-1}} (\Gamma_{0,T}^{0,\hat{\gamma}})^{\frac{1}{1-\alpha}} \\ &= x e^{-\frac{\alpha}{2(1-\alpha)^2} \int_0^T |b_r + \hat{\gamma}_r - \hat{\delta}_r|^2 dr} (N_{0,T}^{0,b-\hat{\delta}})^{\frac{1}{\alpha-1}} (\Gamma_{0,T}^{0,\hat{\gamma}})^{\frac{1}{1-\alpha}}. \end{aligned}$$

It is easy to prove the following proposition.

Proposition 4.6 $(\hat{\delta}, \hat{\gamma})$ may not be unique. But the optimal terminal wealth is unique. Furthermore, we have that for $t \in [0, T]$ and $i = 1, \dots, d$,

$$\begin{cases} \hat{\delta}_{it} = \varepsilon_i \text{ and } \hat{\gamma}_{it} = -K_i, & \text{when } b_{it} > K_i + \varepsilon_i; \\ \hat{\delta}_{it} = -\varepsilon_i \text{ and } \hat{\gamma}_{it} = K_i, & \text{when } b_{it} < K_i - \varepsilon_i; \\ \hat{\delta}_{it} - \hat{\gamma}_{it} = b_{it}, & \text{when } |b_{it}| \leq K_i - \varepsilon_i. \end{cases}$$

Now we employ the dynamic programming principle to calculate the optimal wealth process, the optimal portfolio strategies as well as the utility intensity process.

Suppose that the wealth of an large investor is x at time t . The wealth equation is

$$\begin{cases} dX_s^{t,x} = (\pi'_s b_s - \varepsilon' |\pi_s|) ds + \pi'_s dW_s, & s \in [t, T], \\ X_t^{t,x} = x \geq 0, \end{cases} \quad (4.9)$$

where $\pi \in \bar{\mathcal{A}}(x; t, T)$ (recall (2.9)). The recursive utility is

$$Y_s^{t,x} = u(X_T^{t,x}) - \int_s^T K' |Z_r^{t,x}| dr - \int_s^T Z_r^{t,x'} dW_r, \quad s \in [t, T]. \quad (4.10)$$

Then the dynamic version of our problem is

$$\begin{aligned} & \text{Maximize } Y_t^{t,x}, \\ & s.t. \begin{cases} \pi \in \bar{\mathcal{A}}(x; t, T), \\ (X^{t,x}, \pi^{t,x}) \text{ satisfies Eq.(4.9),} \\ (Y^{t,x}, Z^{t,x}) \text{ satisfies Eq.(4.10),} \end{cases} \end{aligned} \quad (4.11)$$

Define the value function

$$v(t, x) = \sup_{\pi \in \bar{\mathcal{A}}(x)} Y_t^{t,x}.$$

$v(t, x)$ satisfies the following HJB equation (refer to [25]):

$$\begin{cases} \frac{\partial v}{\partial t} + \sup_{\pi \in \mathbb{R}^d} \left[\frac{\partial v}{\partial x} (\pi'_t b_t - \varepsilon' |\pi_t|) + \frac{1}{2} \frac{\partial^2 v}{\partial x^2} \pi'_t \pi_t - K' \left| \frac{\partial v}{\partial x} \pi_t \right| \right] = 0, \\ v(T, x) = u(x) = \frac{1}{\alpha} x^\alpha. \end{cases} \quad (4.12)$$

For any $\gamma \in \mathcal{B}_2$, $(\Gamma_{0,t}^{0,\gamma})$ is a martingale. So we can define a new probability measure P^γ on \mathcal{F}_T via

$$P^\gamma(A) = E[\Gamma_{0,T}^{0,\gamma} I_A], \quad \forall A \in \mathcal{F}_T.$$

Let $E^\gamma[\cdot]$ be the expectation operator with respect to P^γ . Under P^γ , the process

$$W_t^\gamma := W_t - \int_0^t \gamma_s ds, \quad t \in [0, T]$$

is a Brownian motion.

Thus, by the results in section 3, Problem (4.11) is equivalent to the following problem

$$\begin{aligned} & \text{Maximize } E[\Gamma_{t,T}^{0,\hat{\gamma}} u(X^{t,x}(T))] = E^{\hat{\gamma}}[u(X^{t,x}(T))], \\ & s.t. \begin{cases} \pi \in \bar{\mathcal{A}}(x), \\ (X^{t,x}, \pi^{t,x}) \text{ satisfies the following equation (4.14),} \end{cases} \end{aligned} \quad (4.13)$$

$$\begin{cases} dX_s^{t,x} = (\pi'_s b_s - \pi'_s \hat{\delta}_s) ds + \pi'_s dW_s = (\pi'_s b_s + \pi'_s \hat{\gamma}_s - \pi'_s \hat{\delta}_s) ds + \pi'_s dW_s^{\hat{\gamma}} \\ X_t^{t,x} = x \geq 0. \end{cases} \quad (4.14)$$

Therefore, $v(t, x)$ also satisfies the following HJB equation:

$$\begin{cases} \frac{\partial v}{\partial t} + \sup_{\pi \in \mathbb{R}^d} \left[\frac{\partial v}{\partial x} (\pi'_t b_t + \pi'_t \hat{\gamma}_t - \pi'_t \hat{\delta}_t) + \frac{1}{2} \frac{\partial^2 v}{\partial x^2} \pi'_t \pi_t \right] = 0, \\ v(T, x) = u(x) = \frac{1}{\alpha} x^\alpha. \end{cases} \quad (4.15)$$

It is easy to verify the following theorem.

Theorem 4.7 (4.12) and (4.15) share the same solution $v(t, x) = \frac{1}{\alpha} x^\alpha e^{\frac{\alpha}{2(1-\alpha)} \int_t^T (b_r + \hat{\gamma}_r - \hat{\delta}_r)^2 dr}$. For $s \in [t, T]$, the optimal portfolio strategy is

$$\hat{\pi}_s = -\frac{\frac{\partial v}{\partial x}}{\frac{\partial^2 v}{\partial x^2}} (b_s + \hat{\gamma}_s - \hat{\delta}_s) = \frac{1}{1-\alpha} \hat{X}_s^{t,x} (b_s + \hat{\gamma}_s - \hat{\delta}_s),$$

and the utility intensity process is

$$\begin{aligned}\hat{Z}_s &= -\frac{(\frac{\partial v}{\partial x})^2}{\frac{\partial^2 v}{\partial x^2}}(b_s + \hat{\gamma}_s - \hat{\delta}_s) \\ &= \frac{1}{1-\alpha}(\hat{X}_s^{t,x})^\alpha e^{\frac{\alpha}{2(1-\alpha)} \int_s^T (b_r + \hat{\gamma}_r - \hat{\delta}_r)^2 dr} (b_s + \hat{\gamma}_s - \hat{\delta}_s),\end{aligned}$$

where $\hat{X}_s^{t,x}$ is the following optimal wealth process

$$\begin{aligned}\hat{X}_s^{t,x} &= \frac{E[N_{t,T}^{0,b-\hat{\delta}} I(\zeta \frac{N_{t,T}^{0,b-\hat{\delta}}}{\Gamma_{t,T}^{0,\hat{\gamma}}}) | \mathcal{F}_s]}{N_{t,s}^{0,b-\hat{\delta}}} \\ &= \frac{x}{e^{\frac{\alpha}{2(1-\alpha)^2} \int_t^T (b_r + \hat{\gamma}_r - \hat{\delta}_r)^2 dr}} \frac{E[(N_{t,T}^{0,b-\hat{\delta}})^{\frac{\alpha}{\alpha-1}} (\Gamma_{t,T}^{0,\hat{\gamma}})^{\frac{1}{1-\alpha}} | \mathcal{F}_s]}{N_{t,s}^{0,b-\hat{\delta}}} \\ &= \frac{x}{e^{\frac{\alpha}{2(1-\alpha)^2} \int_t^T (b_r + \hat{\gamma}_r - \hat{\delta}_r)^2 dr}} \left(\frac{\Gamma_{t,s}^{0,\hat{\gamma}}}{N_{t,s}^{0,b-\hat{\delta}}} \right)^{\frac{1}{1-\alpha}}.\end{aligned}$$

By this theorem, we know the large investor will invest $\frac{1}{1-\alpha}(b_s + \hat{\gamma}_s - \hat{\delta}_s)$ percent of his wealth to the stocks. Especially, when $|b_{it}| \leq K_i + \varepsilon_i$, the investor will not invest on the i th stock at all.

Remark 4.8 Suppose that all the coefficients are deterministic and $(\bar{\theta}_t)$, $(\underline{\theta}_t)$ are two given d -dimensional processes. Suppose that $\bar{\theta}_t \geq \underline{\theta}_t$, a.e. on $[0, T]$. Then, our method can tackle with the following wealth equation

$$\begin{cases} dX_t = (r_t X_t + \underline{\theta}_t' \pi_t^+ - \bar{\theta}_t' \pi_t^-) dt + \pi_t' dW_t, \\ X_0 = x_0, \quad t \in [0, T] \end{cases} \quad (4.16)$$

where π^+ and π^- denotes the d -dimensional vector with i th component π_i^+ and π_i^- , $i = 1, \dots, d$, respectively. This kind of wealth equation describes the different expected returns for long and short position of the stocks which is appeared in Jouini and Kallal [18] and El Karoui et al [10]. It also appeared in El Karoui et al [9] when there are taxes which must be paid on the gains made on the stocks.

Remark 4.9 When b is a bounded progressively measurable processes, $g(t, z)$ is no longer deterministic. We can prove the quadratic BSDE (4.8) has a unique solution $(\tilde{Y}_t, \tilde{Z}_t)$. There exists a pair $(\hat{\delta}, \hat{\gamma})$ attains the infimum of $g(t, \tilde{Z}_t)$ due to the boundness, closeness and convexity of \mathcal{B}_2 and \mathcal{B}_2' .

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